

A CHARACTERIZATION OF BMO^α -MARTINGALE SPACES BY FRACTIONAL CARLESON MEASURES

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ABSTRACT. We give a characterization of BMO^α -martingale spaces by using fractional Carleson measures. We get the boundedness of martingale transform and square function on BMO^α -martingale spaces easily by using this characterization. We also proved the martingale version of Carleson's inequality related with BMO^α -martingales.

1. INTRODUCTION

In Harmonic Analysis, the space of functions of bounded mean oscillation, or BMO -space, naturally arises as the class of functions whose deviation from their means over cubes is bounded. We recall that a locally integrable function f on \mathbb{R}^n is in $BMO(\mathbb{R})$ -space if $\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx$, where $|Q|$ is the Lebesgue measure of the cube Q in \mathbb{R}^n , f_Q denotes the average (or mean value) of f on Q , and the supremum is taken over all cubes Q in \mathbb{R}^n . It is a good substitution of the L^∞ space, as a dual space of Hardy space H^1 , also in the interpolation theory. Also, Carleson measures are among the most important tools in Harmonic Analysis. A positive measure μ on \mathbb{R}_+^{n+1} is called a Carleson measure if $\|\mu\| = \sup_Q \frac{\mu(Q \times (0, l(Q)))}{|Q|}$, where $l(Q)$ denotes the side length of the cube Q . Fefferman and Stein found that the BMO -space has a natural and deep relationship with Carleson measures, see [3, 4]. In [15], the authors studied the relationship between the function in Morry space and a general kind Carleson measure. This general kind Carleson measure is defined as follows. For $0 < p < 1$, a positive measure μ on \mathbb{R}_+^{n+1} is called a bounded p -Carleson measure if $\|\mu\| = \sup_Q \frac{\mu(Q \times (0, l(Q)))}{|Q|^{1+p}}$.

In this paper we will study an analogue characterization in martingale spaces.

Let (Ω, \mathcal{F}, P) be a complete probability space. For $1 \leq p < \infty$ the usual L_p -space of strong p -integrable scalar-valued functions on (Ω, \mathcal{F}, P) will be denoted by $L^p(\Omega)$ or simply by L^p . Let $\{\mathcal{F}_n\}_{n \geq 0}$ be an increasing sequence of sub- σ -fields of \mathcal{F} such that $\mathcal{F} = \vee \mathcal{F}_n$. We call a sequence $f = \{f_n\}_{n \geq 0}$ in L^1 to be a martingale if $\mathbb{E}(f_{n+1} | \mathcal{F}_n) = f_n$ for every $n \geq 0$. Let $d_n f = f_n - f_{n-1}$ with the convention that $f_{-1} = 0$. $\{d_n f\}_{n \geq 0}$ is the martingale difference sequence of f . To avoid unnecessary convergence problem on infinite series we will assume that all martingales considered in the sequel are finite, unless explicitly stated otherwise. We will adopt the convention that a martingale $f = \{f_n\}_{n \geq 0}$ will be identified with its final value f_∞ whenever the latter exists. And, if $f \in L^1$ we will denote again by f the associated martingale $\{f_n\}$ with $f_n = \mathbb{E}(f | \mathcal{F}_n)$. We refer the reader to [6, 7, 14] for more information on martingale theory.

The main object of this paper is the BMO^α -martingale space given in the following.

Definition 1.1 (BMO^α -martingale space). Let $0 \leq \alpha \leq 1$, $\{\mathcal{F}_n\}_{n \geq 0}$ be an increasing sequence of σ -algebras in a probability space (Ω, \mathcal{F}, P) , $f = \{f_n\}_{n \geq 0}$ an L^2 -martingale relative to $\{\mathcal{F}_n\}_{n \geq 0}$. We

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say that f belongs to BMO^α , if

$$\sup_{n \geq 0} \sup_{A \in \mathcal{F}_n} P(A)^{-1/2-\alpha} \left(\int_A |f - f_{n-1}|^2 dP \right)^{1/2} < \infty.$$

The norm in the space BMO^α is $\|f\|_{BMO^\alpha} = \sup_{n \geq 0} \sup_{A \in \mathcal{F}_n} P(A)^{-1/2-\alpha} \left(\int_A |f - f_{n-1}|^2 dP \right)^{1/2}$.

Note that BMO^0 -martingale space is just the BMO -martingale space.

In fact, if we replace the condition in Definition 1.1 by

$$\sup_{n \geq 0} \sup_{A \in \mathcal{F}_n} P(A)^{-1/p-\alpha} \left(\int_A |f - f_{n-1}|^p dP \right)^{1/p} < \infty,$$

for any $1 \leq p < \infty$ and martingales $f \in L^p$ -martingale space, the space is equivalent with the space defined in Definition 1.1.

Also, we can use another definition of BMO^α -space which is equivalent with Definition 1.1, and we can find it in [6, 7] as in the following.

Let $\mathcal{I}^{(n)}$ be the set of all \mathcal{F}_n -atoms, $n \geq 0$. Denote

$$\omega_n = \sum |I^{(n)}| \chi_{I^{(n)}},$$

where the sum is taken over all $I^{(n)} \in \mathcal{I}^{(n)}$.

Definition 1.2 (BMO^α -martingale spaces). Let $0 \leq \alpha \leq 1$, $\{\mathcal{F}_n\}_{n \geq 0}$ be an increasing sequence of σ -algebras in a probability space (Ω, \mathcal{F}, P) , $f = \{f_n\}_{n \geq 0}$ an L^2 -martingale relative to $\{\mathcal{F}_n\}_{n \geq 0}$. We say that f belongs to BMO^α , if

$$\sup_n \left\| \omega_n^{-\alpha} E(|f - f_{n-1}|^2 | \mathcal{F}_n) \right\|_\infty^{1/2} < \infty.$$

The norm in the space BMO^α -martingale space is $\|f\|_{BMO^\alpha} = \sup_n \left\| \omega_n^{-\alpha} E(|f - f_{n-1}|^2 | \mathcal{F}_n) \right\|_\infty^{1/2}$.

The classical notion of the general Carleson measure in Harmonic Analysis has the following martingale analogue.

Definition 1.3 (Bounded α -Carleson measure). Let μ be a nonnegative measure on $\Omega \times \mathbb{N}$, where \mathbb{N} is equipped with the counting measure dm . μ is called a bounded α -Carleson measure ($0 \leq \alpha < 1$) if

$$\|\mu\|_\alpha := \sup_\tau \frac{\mu(\hat{\tau})}{P(\tau < \infty)^{1+2\alpha}} < \infty,$$

where the supremum runs over all stopping times τ and $\hat{\tau}$ denotes the ‘‘tent’’ over τ :

$$\hat{\tau} = \{(\omega, k) \in \Omega \times \mathbb{N} : k \geq \tau(\omega), \tau(\omega) < \infty\}.$$

When $\alpha = 0$, the α -Carleson measure is just the Carleson measure in martingale theory. In [5], the author studied the relationship between the Carleson measure and vector-valued BMO -martingale space. And for the scalar-valued case, see [7]. In this paper, we will characterize martingales in BMO^α -martingale space in terms of α -Carleson measures. In fact, we have the following.

Theorem 1.4. *The following statements are equivalent:*

- (I) $f \in BMO^\alpha$;
- (II) the measure $|d_k f|^2 dP \otimes dm$ is a bounded α -Carleson measure, i.e.

$$\sup_\tau \frac{1}{P(\tau < \infty)^{1+2\alpha}} \int_{\hat{\tau}} |d_k f|^2 dP \otimes dm < \infty.$$

Related with Carleson measures, there is a famous Carleson’s inequality in Harmonic Analysis which was first proved by Carleson.

Theorem 1.5. For any Carleson measure μ and every μ -measurable function f on \mathbb{R}_+^{n+1} we have

$$\int_{\mathbb{R}_+^{n+1}} |f(x, t)|^p d\mu(x, t) \leq C_n \|\mu\| \int_{\mathbb{R}^n} (f^*(x))^p dx$$

for all $0 < p < \infty$, where $f^*(x) = \sup_{(y, t) \in \{|y-x| < t\}} |f(x, t)|$.

We can get a Carleson's inequality related with BMO^α -martingales. Unlike the inequality related with BMO -martingales, we can get it only for $1 < p < \infty$.

Theorem 1.6. Let $d\mu = \mu_k dP \otimes dm$, with μ_k 's being nonnegative random variables, be a bounded α -Carleson measure ($0 < \alpha < 1$), and $1 < p < \infty$. Then for all adapted processes $f = (f_n)_{n \geq 0}$, we have

$$(1.1) \quad \int_{\Omega \times N} |f_k|^p \mu_k dP \otimes dm \leq \frac{p}{p-1} \|\mu\|_\alpha \|Mf\|_{L^{\frac{1}{2\alpha}}} \|Mf\|_{L^{p-1}}^{p-1},$$

where Mf is the maximal function of $f = (f_n)_{n \geq 0}$. Conversely, if the above inequality, (1.1), holds for some $1 < p < \infty$, with $\frac{p}{p-1} \|\mu\|_\alpha$ replaced by a constant C_p , then μ is a bounded α -Carleson measure and $\|\mu\|_\alpha \leq C_p$.

The paper is organized as follows. In Section 2, we give the proofs of Theorem 1.4 and 1.6. In Section 3, we get the boundedness of the uniformly bounded martingale transform operator, the square function and the maximal operator on BMO^α -martingale spaces easily by using Theorem 1.4.

Throughout this paper, the letter C will denote a positive constant which may change from one instance to another.

2. PROOFS OF THE MAIN THEOREMS

In this section, we will give the proof of Theorem 1.4. In order to do this, we need the following lemma.

Lemma 2.1. Let $f = \{f_n\}_{n \geq 0}$ be an L^2 -martingale. Then

$$\|f\|_{BMO^\alpha} = \sup_{\tau} P(\tau < \infty)^{-1/2-\alpha} \|f - f_{\tau-1}\|_{L^2},$$

where the supremum is taken over all stopping times τ .

Proof. Assume that $\|f\|_{BMO^\alpha} < \infty$ and τ is any stopping time. Then

$$\begin{aligned} P(\tau < \infty)^{-1-2\alpha} \|f - f_{\tau-1}\|_{L^2}^2 &= P(\tau < \infty)^{-1-2\alpha} \sum_{n=1}^{\infty} \int_{\{\tau=n\}} |f - f_{n-1}|^2 dP \\ &\leq \|f\|_{BMO^\alpha}^2 P(\tau < \infty)^{-1-2\alpha} \sum_{n=1}^{\infty} P(\tau = n)^{1+2\alpha} = \|f\|_{BMO^\alpha}^2. \end{aligned}$$

Conversely, let $\beta = \sup_{\tau} P(\tau < \infty)^{-1/2-\alpha} \|f - f_{\tau-1}\|_{L^2}$. For any $n \geq 1$ and $A \in \mathcal{F}_n$, define

$$\tau_A(\omega) = \begin{cases} n, & \omega \in A, \\ \infty, & \omega \notin A. \end{cases}$$

Then,

$$P(A)^{-1-2\alpha} \int_A |f - f_{n-1}|^2 dP = P(\tau_A < \infty)^{-1-2\alpha} \|f - f_{\tau_A-1}\|_{L^2}^2 \leq \beta^2.$$

This proves that $\|f\|_{BMO^\alpha} \leq \beta$. We complete the proof of the lemma. \square

With Lemma 2.1 above, we can give the proof of Theorem 1.4 as follows.

Proof of Theorem 1.4. (I) \Rightarrow (II). Assume that $f \in BMO^\alpha$. For any $1 \leq n \leq m$ and $A \in \mathcal{F}_n$, we have

$$\int_A \sum_{k=n}^m |d_k f|^2 dP \leq C \int_A |f_m - f_{n-1}|^2 dP \leq C \int_A |f - f_{n-1}|^2 dP \leq CP(A)^{1+2\alpha} \|f\|_{BMO^\alpha}^2.$$

This means

$$P(A)^{-1-2\alpha} \int_A \sum_{k=n}^\infty |d_k f|^2 dP \leq C \|f\|_{BMO^\alpha}^2, \quad \text{for any } n \geq 1 \text{ and } A \in \mathcal{F}_n.$$

We define

$$\tau_A(\omega) = \begin{cases} n, & \omega \in A, \\ \infty, & \omega \notin A. \end{cases}$$

Then, since $\Omega \in \mathcal{F}_\tau$ always, we get

$$\begin{aligned} \frac{1}{P(\tau_A < \infty)^{1+2\alpha}} \int_{\widehat{\tau_A}} |d_k f|^2 dP \otimes dm &= \frac{1}{P(\tau_A < \infty)^{1+2\alpha}} \int_\Omega \sum_{k=\tau_A}^\infty |d_k f|^2 \chi_{\{\tau_A < \infty\}} dP \\ &= \frac{1}{P(A)^{1+2\alpha}} \int_A \sum_{k=n}^\infty |d_k f|^2 dP \leq C \|f\|_{BMO^\alpha}^2. \end{aligned}$$

By taking supremum over all stopping times, we know that the measure $|d_k f|^2 dP \otimes dm$ is a bounded α -Carleson measure.

(II) \Rightarrow (I). Let us consider the new σ -fields $\{\mathcal{F}_{k \vee \tau}\}_{k \geq 1}$ and the corresponding martingale \tilde{f} generated by $f - f_\tau$. Then by Doob's stopping time theorem,

$$\tilde{f}_k = \mathbb{E}(f - f_\tau | \mathcal{F}_{k \vee \tau}) = \mathbb{E}(f | \mathcal{F}_{k \vee \tau}) - f_\tau = f_{k \vee \tau} - f_\tau.$$

By Burkholder-Gundy's inequality, we get

$$\begin{aligned} \|f - f_\tau\|_{L^2}^2 &= \|\tilde{f}\|_{L^2}^2 \leq C \left\| \left(\sum_{k=1}^\infty |d_k \tilde{f}|^2 \right)^{1/2} \right\|_{L^2}^2 = C \left\| \left(\sum_{k=1}^\infty |f_{(k+1) \vee \tau} - f_{k \vee \tau}|^2 \right)^{1/2} \right\|_{L^2}^2 \\ &= C \left\| \left(\sum_{k=\tau}^\infty |f_{k+1} - f_k|^2 \right)^{1/2} \right\|_{L^2}^2 = C \left\| \left(\sum_{k=\tau+1}^\infty |d_k f|^2 \right)^{1/2} \chi_{\{\tau < \infty\}} \right\|_{L^2}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|f - f_{\tau-1}\|_{L^2}^2 &\leq C \left(\|f - f_\tau\|_{L^2}^2 + \|f_\tau - f_{\tau-1}\|_{L^2}^2 \right) \\ (2.1) \quad &= C \left\| \left(\sum_{k=\tau}^\infty |d_k f|^2 \right)^{1/2} \chi_{\{\tau < \infty\}} \right\|_{L^2}^2. \end{aligned}$$

Hence, by Lemma 2.1 and (2.1) we obtain

$$\begin{aligned} \|f\|_{BMO^\alpha} &\leq C \sup_\tau P(\tau < \infty)^{-1/2-\alpha} \left\| \left(\sum_{k=\tau}^\infty |d_k f|^2 \right)^{1/2} \chi_{\{\tau < \infty\}} \right\|_{L^2} \\ &= C \sup_\tau \left(\frac{1}{P(\tau < \infty)^{1+2\alpha}} \int_{\widehat{\tau}} |d_k f|^2 dP \otimes dm \right)^{1/2} < \infty. \end{aligned}$$

By the proof above, we know that $\|f\|_{BMO^\alpha} \sim \sup_\tau \left(\frac{1}{P(\tau < \infty)^{1+2\alpha}} \int_{\widehat{\tau}} |d_k f|^2 dP \otimes dm \right)^{1/2}$. \square

In the following, we will give the proof of Theorem 1.6.

Proof of Theorem 1.6. First, assume that μ is a bounded α -Carleson measure, $f = (f_n)_{n \geq 0}$ is adapted. For any $\lambda > 0$, define the stopping time $\tau = \inf\{n : |f_n| > \lambda\}$. Then we have

$$Mf(\omega) = \chi_{\{\tau < \infty\}}(\omega) \text{ and } \{(\omega, k) : |f_k| > \lambda\} \subset \{(\omega, k) : k \geq \tau(\omega), \tau(\omega) < \infty\}.$$

Hence,

$$\begin{aligned} \int_{\Omega \times N} |f_k|^p \mu_k dP \otimes dm &= p \int_0^\infty \lambda^{p-1} \mu(\{(\omega, k) : |f_k| > \lambda\}) d\lambda \\ &\leq p \int_0^\infty \lambda^{p-1} \mu(\{(\omega, k) : k \geq \tau(\omega), \tau(\omega) < \infty\}) d\lambda \\ &\leq \|\mu\|_\alpha p \int_0^\infty \lambda^{p-1} P(\tau < \infty)^{1+2\alpha} d\lambda \\ &= \|\mu\|_\alpha p \int_0^\infty \lambda (P(\{Mf > \lambda\}))^{2\alpha} \lambda^{p-2} P(\{Mf > \lambda\}) d\lambda \\ &\leq \|\mu\|_\alpha \|Mf\|_{L^{\frac{1}{2\alpha}}, \infty} \cdot p \int_0^\infty \lambda^{p-2} P(\{Mf > \lambda\}) d\lambda \\ &\leq \frac{p}{p-1} \|\mu\|_\alpha \|Mf\|_{L^{\frac{1}{2\alpha}}} \|Mf\|_{L^{p-1}}. \end{aligned}$$

Thus we proved the inequality (1.1). Reversely, we assume that (1.1) holds for some $1 < p < \infty$ with a constant C_p . For any stopping time τ , let $f_n(\omega) = \chi_{\{\tau \leq n\}}(\omega)$, for any $\omega \in \Omega$, $n \geq 0$. Then $Mf(\omega) = \chi_{\{\tau < \infty\}}(\omega)$, and

$$\int_{\Omega \times N} \chi_{\{\tau \leq k\}} \mu_k dP \otimes dm = \int_{\Omega \times N} |f_k|^p \mu_k dP \otimes dm \leq C_p \|Mf\|_{L^{\frac{1}{2\alpha}}} \|Mf\|_{L^{p-1}}^{p-1} = C_p |\{\tau < \infty\}|^{1+2\alpha}.$$

This means that $\|\mu\|_\alpha \leq C_p$. We complete the proof. \square

Remark 2.2. In particular, if $d\mu = |d_k f|^2 dP \otimes dm$ with $f = (f_n)_{n \geq 0}$ is a martingale, then, combined with Theorem 1.4, we have that $f \in BMO^\alpha$, if and only if

$$\int_{\Omega \times N} |f_k|^p |d_k f|^2 dP \otimes dm = \int_{\Omega \times N} |f_k|^p d\mu \leq C_p \|Mf\|_{L^{\frac{1}{2\alpha}}} \|Mf\|_{L^{p-1}}^{p-1},$$

for all adapted $f = (f_n)_{n \geq 0}$ and some $1 < p < \infty$.

3. APPLICATIONS

In this section, we will give some applications of our main theorem. In order to adapt our result to the applications we need the following remark.

Remark 3.1. Theorem 1.4 can also be stated in a Hilbert-valued setting. If f takes value in a Hilbert space \mathbb{H} and the absolute values in the statements are replaced by the norm in \mathbb{H} , then the result holds also.

For more information about the vector-valued martingales, we refer the reader to [6, 9, 10, 11].

3.1. The boundedness of martingale transform on BMO^α -martingale spaces.

Theorem 3.2. Let $\{v_n\}_{n \geq 0}$ be a uniformly bounded \mathcal{F}_n -predictable sequence. Then the martingale transform given by

$$(Tf)_n = \sum_{k=0}^n v_k d_k f$$

is bounded on BMO^α -martingale spaces.

Proof. For any $f \in BMO^\alpha$, by Theorem 1.4, we have that for any stopping time τ ,

$$\frac{1}{P(\tau < \infty)^{1+2\alpha}} \int_{\hat{\tau}} |d_k f|^2 dP \otimes dm \leq C \|f\|_{BMO^\alpha}^2.$$

Since $\{v_k\}_{k \geq 0}$ is uniformly bounded, then for any stopping time τ ,

$$\begin{aligned} \frac{1}{P(\tau < \infty)^{1+2\alpha}} \int_{\hat{\tau}} |d_k(Tf)|^2 dP \otimes dm &= \frac{1}{P(\tau < \infty)^{1+2\alpha}} \int_{\hat{\tau}} |v_k d_k f|^2 dP \otimes dm \\ &\leq C \cdot \frac{1}{P(\tau < \infty)^{1+2\alpha}} \int_{\hat{\tau}} |d_k f|^2 dP \otimes dm \leq C \|f\|_{BMO^\alpha}^2. \end{aligned}$$

Then by Theorem 1.4, we get $Tf \in BMO^\alpha$ and $\|Tf\|_{BMO^\alpha} \leq C \|f\|_{BMO^\alpha}$. \square

3.2. The boundedness of square function on BMO^α -martingale spaces.

Theorem 3.3. *The square function $S(f) = \left(\sum_{k=0}^{\infty} |d_k f|^2 \right)^{1/2}$ is bounded on BMO^α -martingale spaces.*

Proof. Let us consider the ℓ^2 -valued martingale transform:

$$Uf = \sum_{k=1}^{\infty} v_k d_k f \quad \text{and} \quad U_n f = \sum_{k=1}^n v_k d_k f$$

with $v_k = (0, \dots, 0, 1, 0, \dots)$ for any $k \geq 1$. It maps a \mathbb{R} -valued martingale into an ℓ^2 -valued martingale. So, for any $f \in BMO^\alpha$, by Theorem 1.4,

$$\frac{1}{P(\tau < \infty)^{1+2\alpha}} \int_{\hat{\tau}} \|d_k(Uf)\|_{\ell^2}^2 dP \otimes dm = \frac{1}{P(\tau < \infty)^{1+2\alpha}} \int_{\hat{\tau}} |d_k f|^2 dP \otimes dm \leq C \|f\|_{BMO^\alpha}^2.$$

Therefore, by Remark 3.1, we have $Uf \in BMO_{\ell^2}^\alpha$ and $\|U(f)\|_{BMO_{\ell^2}^\alpha} \leq C \|f\|_{BMO^\alpha}$. Since, for any $n \geq 1$,

$$\|U_n f\|_{\ell^2} = \left(\sum_{k=1}^n |d_k f|^2 \right)^{1/2} = S_n(f),$$

we have, for any $A \in \mathcal{F}_n$,

$$\begin{aligned} P(A)^{-1-2\alpha} \int_A |S(f) - S_{n-1}(f)|^2 dP &= P(A)^{-1-2\alpha} \int_A \|\|Uf\|_{\ell^2} - \|U_{n-1}(f)\|_{\ell^2}\|^2 dP \\ &\leq P(A)^{-1-2\alpha} \int_A \|U(f) - U_{n-1}(f)\|_{\ell^2}^2 dP \leq C \|U(f)\|_{BMO_{\ell^2}^\alpha}^2. \end{aligned}$$

Whence, $\|S(f)\|_{BMO^\alpha} \leq C \|U(f)\|_{BMO_{\ell^2}^\alpha} \leq C \|f\|_{BMO^\alpha}$. \square

3.3. The boundedness of maximal function on BMO^α -martingale spaces.

Theorem 3.4. *The maximal function $M(f) = \sup_{n \geq 0} |f_n|$ is bounded on BMO^α -martingale spaces.*

The proof of Theorem 3.4 is similar to the proof of the boundedness of maximal function on BMO -martingale spaces with small variations as in [7].

3.4. UMD Banach lattice. In [2], Burkholder considered the UMD spaces as in the following.

Definition 3.5. A Banach space \mathbb{B} is said to be a UMD space if for all \mathbb{B} -valued martingales $\{f_n\}_{n \geq 0}$ and all sequences $\{\varepsilon_n\}_{n \geq 0}$ with $\varepsilon_n = \pm 1$,

$$\|\varepsilon_0 d_0 f + \dots + \varepsilon_k d_k f\|_{L_{\mathbb{B}}^p} \leq C_p \|d_0 f + \dots + d_k f\|_{L_{\mathbb{B}}^p}, \quad \text{for all } k \geq 0,$$

where $1 < p < \infty$ and the constant $C_p > 0$ is independent of k .

It is known that the existence of one p_0 satisfying the inequality is enough to assure the existence of the rest of p , $1 < p < \infty$. In this section, we concentrate on the Banach lattice. Let \mathbb{B} be a Banach lattice. Without loss of generality we assume that \mathbb{B} is a Banach lattice of measurable functions on measure space (Ω, dP) . We refer the reader to [8] for more information on Banach lattices.

By Theorem 3.4 in [5] and our characterization theorem of BMO^α -martingales in Theorem 1.4, we can get a characterization of UMD Banach lattice in the following theorem.

Theorem 3.6. *Let \mathbb{B} be a Banach lattice. The following statements are equivalent:*

- (I) \mathbb{B} is a UMD space;
- (II) there exists a constant $C > 0$ such that

$$C^{-1} \|f\|_{BMO_{\mathbb{B}}^\alpha} \leq \sup_{\tau} \left(\frac{1}{P(\tau < \infty)^{1+2\alpha}} \int_{\hat{\tau}} \|d_k f\|_{\mathbb{B}}^2 dP \otimes dm \right)^{1/2} \leq C \|f\|_{BMO_{\mathbb{B}}^\alpha},$$

for any \mathbb{B} -valued martingale f .

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